

FLUCTUATION LIMITS OF A LOCALLY REGULATED POPULATION AND GENERALIZED LANGEVIN EQUATIONS

MLADEN SAVOV AND SHI-DONG WANG

ABSTRACT. We consider a locally regulated spatial population model introduced by Bolker and Pacala. Based on the deterministic approximation studied by Fournier and Méléard, we prove that the fluctuation theorem holds under some mild moment conditions. The limiting process is shown to be an infinite-dimensional Gaussian process solving a generalized Langevin equation. In particular, we further consider its properties in one-dimension case, which is characterized as a time-inhomogeneous Ornstein-Uhlenbeck process.

1. INTRODUCTION

It is well known that branching processes have been widely used to model the evolution in biological populations. If, in addition, the individuals are assumed to follow some independent motions (like Brownian motion or random walks), the system can be approximated by the so-called Dawson-Watanabe superprocess (refer to [6, 8, 17]). The most common feature of these processes is that branching and spatial motion are independent.

Since individuals can reproduce, mutate and die in varying rates according to their different spatial characteristics (phenotypes), one reasonable improvement we can make is to add spatial components to both branching and dispersal parameters. Nevertheless, the spatial-dependent components destroy the independence between branching and dispersal while bringing us abundant information from the phenotypic point of view, and even though, the model is still deficient: such as in the finite-dimensional branching process model, the populations either die out or escape to infinity, depending on the mean matrix of the offspring distribution. The model thus can not predict a non-trivial equilibrium which actually happens quite often in the biological world. Bolker and Pacala [2] propose a self-regulated model which attains the above two improved features. By employing the idea of the ordinary logistic growth equation, they introduce a competition term in the density-dependent populations, which can help the system to attain equilibria under specific conditions. However, the loss of branching property can also cause some new technical difficulties when we study some properties such as weak convergence from branching particle systems to a continuum limit.

Law and Dieckmann [16] study this model in parallel with Bolker and Pacala [2]. We simply call it BPDFL model. In recent years, this model has been extensively studied in papers such as Etheridge [9], Fournier and Méléard [11], Champagnat [4], Lambert [15], Dawson and Greven [7]. Etheridge [9] studies two diffusion limits, one is a *stepping stone version* of the BPDFL model (interacting diffusions indexed by \mathbb{Z}^d) and another is a *superprocess version* of it. In that paper, sufficient conditions are given for survival and local extinction. Fournier and Méléard [11] formulate a pathwise construction of the BPDFL process in terms of Poisson point processes.

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Under the finiteness of third moment condition, they rigorously obtain a deterministic approximation (law of large numbers) of the BPDF processes. Our work is based on the formalization of Fournier and Méléard [11]. In the papers Champagnat [4], Champagnat and Méléard [5], Dawson and Greven [7], they investigate long term behaviour of respective populations by the method of multiple time scales analysis.

In this paper we aim to present and prove the fluctuation theorem in a general framework set by Fournier and Méléard [11], which could be applied in the derivative models studied by the referred authors. As for a sequence of density-dependent population processes with only finite-many types, Kurtz [14] proves its central limit theorem, which is characterized by some finite-dimensional diffusion process. As for infinite-dimensional population models, Gorostiza and Li [12] prove the high-density fluctuations of a branching particle system with immigration, where they use the classical Laplace transform method owing to the branching property. In our case, this approach doesn't work anymore due to the loss of branching property.

The remainder of the paper is structured as follows. In Section 2, we briefly describe the model and give some preliminary results. More precisely, we recall the law of large numbers of the BPDF processes proved by Fournier and Méléard [11]. In Section 3, we build the fluctuation theorem and prove the tightness and finite-dimensional convergence based on some moment estimates in subsequent sections. In Section 4, in order to better understand the limiting process, we show it to be the solution of an infinite-dimensional inhomogeneous Langevin equation, which can be viewed as evolving in a deterministic medium. In Section 5, we consider a degenerate case, the one dimensional version of the fluctuation limit. A precise characterization of the fluctuation diffusion is given as a time-inhomogeneous Ornstein-Uhlenbeck process. We study its stationary distribution as well.

2. MODEL

2.1. Notation and description of the process. Following [2], we assume that the population at time t is composed of a finite number $I(t)$ of individuals characterized by their phenotypic traits $x_1(t), \dots, x_{I(t)}(t)$ taking values in a compact subset \mathcal{X} of \mathbb{R}^d .

We denote by $\mathcal{M}_F(\mathcal{X})$ the set of finite measures on \mathcal{X} (including negative-valued measures). Let $\mathcal{M}_a(\mathcal{X}) \subset \mathcal{M}_F(\mathcal{X})$ be the set of counting measures on \mathcal{X} :

$$\mathcal{M}_a(\mathcal{X}) = \left\{ \sum_{i=1}^n \delta_{x_i} : x_1, \dots, x_n \in \mathcal{X}, n \in \mathbb{N} \right\}.$$

Then, the population process can be represented as:

$$\nu_t = \sum_{i=1}^{I(t)} \delta_{x_i(t)}.$$

Let $\mathcal{B}(\mathcal{X})$ denote the totality of functions on \mathcal{X} that are bounded measurable. Let $C^\infty(\mathcal{X})$ denote the space of infinitely differentiable functions on \mathbb{R}^d with support contained in \mathcal{X} . Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of (infinitely differentiable, rapidly decreasing) testing functions on \mathbb{R}^d whose topological dual space is $\mathcal{S}'(\mathbb{R}^d)$, and $\langle \cdot, \cdot \rangle$ the canonical bilinear form on $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. When $\mu \in \mathcal{S}'(\mathbb{R}^d)$ is a (signed) measure, then $\langle \mu, \phi \rangle = \int \phi d\mu$, $\phi \in \mathcal{S}(\mathbb{R}^d)$. With a slight abuse of notation we denote by $\mathcal{S}'(\mathcal{X}) \subset \mathcal{S}'(\mathbb{R}^d)$ the subset of tempered distributions $\psi \in \mathcal{S}'(\mathbb{R}^d)$ which satisfy $\langle \phi, \psi \rangle = 0$, for any $\phi \in \mathcal{S}(\mathcal{X}^c)$, i.e. $\text{Supp } \phi \cap \mathcal{X} = \emptyset$. Note that $\mathcal{M}_F(\mathcal{X}) \subset \mathcal{S}'(\mathcal{X})$ which follows immediately from the definition of $\mathcal{S}(\mathcal{X})$.

Let's specify the population processes $(\nu_t^n)_{t \geq 0}$ by introducing a sequence of biological parameters, for $n \in \mathbb{N}$:

- $b_n(x)$ is the rate of birth from an individual with trait x .
- $d_n(x)$ is the rate of death of an individual with trait x because of “aging”.
- $\alpha_n(x, y)$ is the competition kernel felt by some individual with trait x from another individual with trait y .
- $D_n(x, dz)$ is the children’s dispersion law from the mother with trait x . Assume that

$$D_n(x, dz) = m_n(x, z)dz. \quad (2.1)$$

Here, $m_n(x, z)$ is the probability density for mutation variation, which satisfies

$$\int_{z \in \mathbb{R}^d, x+z \in \mathcal{X}} m_n(x, z)dz = 1.$$

Fournier and Méléard [11] have formulated a pathwise construction of the BPDFL process $\{(\nu_t^n)_{t \geq 0}; n \in \mathbb{N}\}$ in terms of Poisson random measures and justified its infinitesimal generator defined for any $\Phi \in \mathcal{B}(\mathcal{M}_a(\mathcal{X}))$:

$$\begin{aligned} L_0^n \Phi(\nu) = & \int_{\mathcal{X}} \nu(dx) \int_{\mathbb{R}^d} \left(\Phi(\nu + \delta_{x+z}) - \Phi(\nu) \right) b_n(x) D_n(x, dz) \\ & + \int_{\mathcal{X}} \nu(dx) \left(\Phi(\nu - \delta_x) - \Phi(\nu) \right) \left(d_n(x) + \int_{\mathcal{X}} \alpha_n(x, y) \nu(dy) \right). \end{aligned} \quad (2.2)$$

The first term is used to model birth events, while the second term which is nonlinear is interpreted as natural death and competing death.

Instead of studying the original BPDFL processes defined by (2.2), our goal is to study the rescaled processes

$$X_t^n := \frac{\nu_t^n}{n}, \quad t \geq 0 \quad (2.3)$$

since it provides us a macroscopic approximation when we take the large population limits (we will see later, the initial population is proportional to n in some sense). The infinitesimal generator of the rescaled BPDFL process has the form, for any $\Phi \in \mathcal{B}(\mathcal{M}_F(\mathcal{X}))$:

$$\begin{aligned} L^n \Phi(\mu) = & \int_{\mathcal{X}} n\mu(dx) \int_{\mathbb{R}^d} \left(\Phi(\mu + \frac{\delta_{x+z}}{n}) - \Phi(\mu) \right) b_n(x) D_n(x, dz) \\ & + \int_{\mathcal{X}} n\mu(dx) \left(\Phi(\mu - \frac{\delta_x}{n}) - \Phi(\mu) \right) \left(d_n(x) + \int_{\mathcal{X}} \alpha_n(x, y) n\mu(dy) \right). \end{aligned} \quad (2.4)$$

2.2. Preliminary results. Let’s denote by (A) the following assumptions:

(A1) There exist $b(x), d(x), f_i(x), g_i(x) \in C^\infty(\mathcal{X})$ ($1 \leq i \leq m^d$) and $m(x, z) \in C^\infty(\mathcal{X} \times (\mathcal{X} - \mathcal{X}))$ such that, for $x, y \in \mathcal{X}, z \in \mathcal{X} - \mathcal{X}, n \in \mathbb{N}$,

$$0 < b_n(x) \equiv b(x), \quad 0 < d_n(x) \equiv d(x), \quad m_n(x, z) \equiv m(x, z),$$

$$0 < \alpha_n(x, y) = \frac{\alpha(x, y)}{n}, \quad \alpha(x, y) = \sum_{i=1}^{m^d} f_i(x) g_i(y).$$

(A2) $b(x) - d(x) > 0$.

The first assumption implies that there exist constants $\bar{b}, \bar{d}, \bar{\alpha}$ such that $b(x) \leq \bar{b}, d(x) \leq \bar{d}, \alpha(x, y) \leq \bar{\alpha}$. The assumption on $\alpha(x, y)$ being the sum above is purely technical for the proof of Lemma 3.8 and the choice of m^d irrelevant as it can be any positive integer. It seems that the technical restriction on α is very hard to remove at the level of CLT when there is an interaction. However, one easily sees that any smooth function α can be approximated in the

supremum norm by expressions of α as in (A1). This then is perfectly suitable for any practical or numerical purpose.

By neglecting the high order moment, Bolker and Pacala [2] use the “moment closure” procedure to approximate the stochastic population processes. As we can see from the generator form (2.4), it should be enough to “close” the second order moment due to the quadratic nonlinear term. Actually the result proved by Fournier and Méléard still holds under a second moment condition $\sup_{n \geq 1} \mathbb{E}\langle X_0^n, 1 \rangle^2 < \infty$. We only recall their result here without mentioning the detailed proof repeatedly.

Theorem 2.1 (Convergence to a nonlinear integro-differential equation). *Under the assumption (A1) consider the sequence of processes $(X_t^n)_{t \geq 0}$ defined in (2.3). Suppose that (X_0^n) converges in law to some deterministic finite measure $X_0 \in \mathcal{M}_F(\mathcal{X})$ as $n \rightarrow \infty$ and satisfies $\sup_{n \geq 1} \mathbb{E}\langle X_0^n, 1 \rangle^2 < \infty$.*

Then the sequence of processes $(X_t^n)_{t \geq 0}$ converges in law as $n \rightarrow \infty$, on $\mathbb{D}([0, \infty), \mathcal{M}_F(\mathcal{X}))$, to a deterministic measure-valued process $(X_t)_{t \geq 0} \in \mathbb{C}([0, \infty), \mathcal{M}_F(\mathcal{X}))$, where $(X_t)_{t \geq 0}$ is the unique solution satisfying

$$\begin{aligned} \langle X_t, \phi \rangle = & \langle X_0, \phi \rangle + \int_0^t ds \int_{\mathcal{X}} X_s(dx) b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \\ & - \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \right). \end{aligned} \quad (2.5)$$

for any $\phi \in \mathcal{B}(\mathcal{X})$ and

$$\sup_{t \in [0, T]} \langle X_t, 1 \rangle < \infty. \quad (2.6)$$

Finally, it comes to a natural question: how does $(X_t^n)_{t \geq 0}$ fluctuate around the macroscopic limit $(X_t)_{t \geq 0}$ given above? A natural candidate to be investigated could be the centralized sequence of processes:

$$Y_t^n := \frac{\nu_t^n - nX_t}{\sqrt{n}} = \sqrt{n}(X_t^n - X_t). \quad (2.7)$$

In the following proposition, we will give some martingale properties of the processes $(Y_t^n)_{t \geq 0}$, which will play a key role in the proof of the main theorem.

Proposition 2.2. *Admit the same assumptions as in Theorem 2.1. For fixed $n \in \mathbb{N}$ and $\phi \in \mathcal{B}(\mathcal{X})$, the process*

$$\begin{aligned} M_t^n(\phi) := & \langle Y_t^n, \phi \rangle - \langle Y_0^n, \phi \rangle - \int_0^t ds \int_{\mathcal{X}} Y_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \\ & + \int_0^t ds \int_{\mathcal{X}} \phi(x) d(x) Y_s^n(dx) \\ & + \sqrt{n} \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \\ & - \sqrt{n} \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \end{aligned} \quad (2.8)$$

is a càdlàg square integrable martingale with quadratic variation

$$\begin{aligned} \langle M^n(\phi) \rangle_t &= \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi^2(x+z) D(x, dz) \\ &\quad + \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) \phi^2(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right). \end{aligned} \quad (2.9)$$

Proof. Recall the generator (2.4), for bounded measurable functional Φ on $\mathcal{M}_F(\mathcal{X})$, the process

$$\Phi(X_t^n) - \Phi(X_0^n) - \int_0^t L^n \Phi(X_s^n) ds$$

is a càdlàg square integrable martingale. If we take $\Phi(\mu) = \langle \mu, \phi \rangle$, $\forall \phi \in \mathcal{B}(\mathcal{X})$, one obtains that

$$\begin{aligned} N_t^n(\phi) &:= \langle X_t^n, \phi \rangle - \langle X_0^n, \phi \rangle - \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \\ &\quad + \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) \phi(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right) \end{aligned} \quad (2.10)$$

is a càdlàg martingale. By applying Itô's formula to $\langle X_t^n, \phi \rangle^2$, we have

$$\begin{aligned} \langle X_t^n, \phi \rangle^2 - \langle X_0^n, \phi \rangle^2 &- 2 \int_0^t ds \langle X_s^n, \phi \rangle \int_{\mathcal{X}} X_s^n(dx) \left\{ b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \right. \\ &\quad \left. - \phi(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right) \right\} - \langle N_t^n(\phi) \rangle_t \end{aligned}$$

is a martingale. On the other hand, if we take $\Phi(\mu) = \langle \mu, \phi \rangle^2$, it follows that

$$\begin{aligned} \langle X_t^n, \phi \rangle^2 - \langle X_0^n, \phi \rangle^2 &- \int_0^t L^n \Phi(X_s^n) ds \\ &= \langle X_t^n, \phi \rangle^2 - \langle X_0^n, \phi \rangle^2 - 2 \int_0^t ds \langle X_s^n, \phi \rangle \int_{\mathcal{X}} X_s^n(dx) \left\{ b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \right. \\ &\quad \left. - \phi(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right) \right\} \\ &\quad - \frac{1}{n} \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi^2(x+z) D(x, dz) \\ &\quad - \frac{1}{n} \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) \phi^2(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right) \end{aligned} \quad (2.11)$$

is a martingale. By comparing the two decompositions of the semimartingale $\langle X_t^n, \phi \rangle^2$ above, one obtains that

$$\begin{aligned} \langle N_t^n(\phi) \rangle_t &= \frac{1}{n} \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi^2(x+z) D(x, dz) \\ &\quad + \frac{1}{n} \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) \phi^2(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right). \end{aligned} \quad (2.12)$$

Owing to (2.10) and (2.5), do the operation $(\langle X_t^n, \phi \rangle - \langle X_t, \phi \rangle)$ and let $M_t^n(\phi) := \sqrt{n} N_t^n(\phi)$, to conclude the proof by the definition of (Y_t^n) in (2.7). \square

3. FLUCTUATION THEOREM

In this section, our aim is to study the asymptotic behavior of $(Y_t^n)_{t \geq 0}$ as $n \rightarrow \infty$. The following theorem, the main result of the paper, shows that $(Y_t^n)_{t \geq 0}$ indeed converges to the unique solution of a martingale problem.

In the following sections, we will use the notation given in Section 2 without declaration. We always assume that assumption (A1) holds.

Theorem 3.1. *Admit assumption (A1) and suppose that there exists a deterministic finite nonnegative measure $X_0 \in \mathcal{M}_F(\mathcal{X})$ such that $Y_0^n = \sqrt{n}(X_0^n - X_0)$ satisfies for some $\delta > 0$*

$$\sup_{n \geq 1} \left(\sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} |\langle Y_0^n, \phi \rangle|^{4+\delta} \right) < \infty. \quad (3.1)$$

Suppose that (Y_0^n) converges in law to a finite (maybe random) measure γ as $n \rightarrow \infty$. Then, the process $(Y_t^n)_{t \geq 0}$ converges in law as $n \rightarrow \infty$ on $\mathbb{D}([0, \infty), \mathcal{S}'(\mathcal{X}))$ to a process $(Y_t)_{t \geq 0} \in \mathbb{C}([0, +\infty), \mathcal{S}'(\mathcal{X}))$ where $(Y_t)_{t \geq 0}$ is the unique solution satisfying

$$\begin{aligned} \langle Y_t, \phi \rangle &= \langle \gamma, \phi \rangle + \int_0^t ds \left\langle Y_s, b(\cdot) \int_{\mathbb{R}^d} \phi(\cdot + z) D(\cdot, dz) \right\rangle \\ &\quad - \int_0^t ds \left\langle Y_s, d(\cdot) \phi(\cdot) \right\rangle \\ &\quad - \int_0^t ds \left\langle Y_s, \int_{\mathcal{X}} \alpha(x, \cdot) \phi(x) X_s(dx) \right\rangle \\ &\quad - \int_0^t ds \left\langle Y_s, \phi(\cdot) \int_{\mathcal{X}} \alpha(\cdot, y) X_s(dy) \right\rangle \\ &\quad + M_t(\phi) \end{aligned} \quad (3.2)$$

for any $\phi \in \mathcal{S}(\mathbb{R}^d)$. Here, $(X_t)_{t \geq 0}$ is the solution defined by the deterministic nonlinear equation (2.5), while $M_t(\phi)$ is a continuous martingale with quadratic variation

$$\begin{aligned} \langle M_t(\phi) \rangle_t &= \int_0^t ds \int_{\mathcal{X}} X_s(dx) b(x) \int_{\mathbb{R}^d} \phi^2(x + z) D(x, dz) \\ &\quad + \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi^2(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \right). \end{aligned} \quad (3.3)$$

Remark 3.2. *The argument above makes essentially use of the initial moment (3.1) and the initial convergence condition. This condition fulfils the assumptions needed in Theorem 2.1. Therefore, the law of large number limit $(X_t)_{t \geq 0}$ is well defined (see Lemma 3.4).*

Remark 3.3. *To avoid confusion, let us point out that we first prove that Y is an $\mathcal{S}'(\mathbb{R}^d)$ -valued process, and $\mathcal{M}_F(\mathcal{X}) \subset \mathcal{S}'(\mathbb{R}^d)$. Subsequently, we show that Y is an $\mathcal{S}'(\mathcal{X})$ -valued process. Therefore, there is no special argument related to the structural properties of $\mathcal{S}'(\mathcal{X})$. Note that Méléard [18] studies the convergence of fluctuations associated with Boltzmann equations in a weighted Sobolev space with a “Sobolev embedding” technique.*

Proving the theorem is the content of the following sections.

3.1. Moment estimates and tightness. The tightness criterion is established for semimartingales based on the moment estimates (see [8]). Our first two lemmas give the uniform second order moment estimates for a sequence of processes over finite time intervals.

Lemma 3.4. *Suppose that a sequence of random variables (Y_0^n) in $\mathcal{M}_F(\mathcal{X})$ satisfies the same condition as in Theorem 3.1. Then, $X_0^n \xrightarrow{\text{in law}} X_0$, as $n \rightarrow \infty$, and*

$$\sup_{n \geq 1} \left(\sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} \langle Y_0^n, \phi \rangle^2 \right) < \infty \quad (3.4)$$

and

$$\sup_{n \geq 1} \mathbb{E} \langle X_0^n, 1 \rangle^2 < \infty. \quad (3.5)$$

Hence, Theorem 2.1 holds.

In particular, for any $T < \infty$, there exists a constant $C_T^{(1)} > 0$ such that

$$\sup_{n \geq 1} \mathbb{E} \left[\sup_{0 \leq t \leq T} \langle X_t^n, 1 \rangle^2 \right] \leq C_T^{(1)}. \quad (3.6)$$

Proof. In fact, the convergence from (X_0^n) to X_0 in law can be implied by the convergence from (Y_0^n) to γ . By Hölder inequality, we easily get that

$$\sup_{n \geq 1} \left(\sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} \langle Y_0^n, \phi \rangle^2 \right) \leq \sup_{n \geq 1} \left(\sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} |\langle Y_0^n, \phi \rangle|^{4+\delta} \right)^{\frac{2}{4+\delta}} < \infty. \quad (3.7)$$

On the other hand, because of the definition of (Y_0^n) as in (2.7), we obtain that

$$\sup_{n \geq 1} \mathbb{E} \langle X_0^n, 1 \rangle^2 \leq 2 \langle X_0, 1 \rangle^2 + 2 \sup_{n \geq 1} \frac{1}{n} \mathbb{E} \langle Y_0^n, 1 \rangle^2 < \infty.$$

Now the proof of the moment estimate (3.6) follows immediately from [11, Theorem 3.1] by applying the Gronwall's lemma. \square

Lemma 3.5. *Suppose that a sequence of random variables $Y_0^n \in \mathcal{M}_F(\mathcal{X})$ satisfies (3.1). Then, for any $T < \infty$, there exists a constant $C_T^{(2)} > 0$ such that*

$$\sup_{n \geq 1} \left(\sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} \left[\sup_{0 \leq t \leq T} \langle Y_t^n, \phi \rangle^2 \right] \right) \leq C_T^{(2)}. \quad (3.8)$$

Proof. From Proposition 2.2, by Hölder inequality, one obtains that

$$\begin{aligned}
\langle Y_t^n, \phi \rangle^2 &\leq 2 \left\{ \langle Y_0^n, \phi \rangle^2 + t \int_0^t \left(\int_{\mathcal{X}} Y_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \right)^2 ds \right. \\
&\quad + t \int_0^t \left(\int_{\mathcal{X}} Y_s^n(dx) d(x) \phi(x) \right)^2 ds \\
&\quad + nt \int_0^t \left(\int_{\mathcal{X}} X_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right. \\
&\quad \quad \left. - \int_{\mathcal{X}} X_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right)^2 ds \\
&\quad \left. + [M_t^n(\phi)]^2 \right\} \\
&\leq 2 \left\{ \langle Y_0^n, \phi \rangle^2 + t \int_0^t \left(\int_{\mathcal{X}} Y_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \right)^2 ds \right. \\
&\quad + t \int_0^t \left(\int_{\mathcal{X}} Y_s^n(dx) d(x) \phi(x) \right)^2 ds \\
&\quad + 2t \int_0^t \left(\int_{\mathcal{X}} Y_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right)^2 ds \\
&\quad \quad + 2t \int_0^t \left(\int_{\mathcal{X}} X_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s^n(dy) \right)^2 ds \\
&\quad \left. + [M_t^n(\phi)]^2 \right\}. \tag{3.9}
\end{aligned}$$

Since $\mathbb{E}[\sup_{0 \leq t \leq T} \langle Y_t^n, \phi \rangle^2]$ is a finite quantity thanks to X and the definition of Y^n , for any fixed $T < \infty$, first take the supremum over time interval $[0, T]$, then take expectations on both sides. It follows that, for any $\phi \in \mathcal{B}(\mathcal{X})$ satisfying $\|\phi\|_\infty \leq 1$,

$$\begin{aligned}
\mathbb{E}[\sup_{0 \leq t \leq T} \langle Y_t^n, \phi \rangle^2] &\leq 2\mathbb{E}\langle Y_0^n, \phi \rangle^2 + 2T\bar{b}^2 \int_0^T \mathbb{E} \left(\int_{\mathcal{X}} Y_s^n(dx) \frac{b(x)}{\bar{b}} \int_{\mathcal{X}} \phi(x+z) D(x, dz) \right)^2 ds \\
&\quad + 2T\bar{d}^2 \int_0^T \mathbb{E} \left(\int_{\mathcal{X}} Y_s^n(dx) \frac{d(x)}{\bar{d}} \phi(x) \right)^2 ds \\
&\quad + 4T\bar{\alpha}^2 \int_0^T \mathbb{E} \left(\int_{\mathcal{X}} Y_s^n(dx) \phi(x) \int_{\mathcal{X}} \frac{\alpha(x, y)}{\bar{\alpha}} X_s^n(dy) \right)^2 ds \\
&\quad + 4T\bar{\alpha}^2 \int_0^T \mathbb{E} \left(\int_{\mathcal{X}} X_s^n(dx) \phi(x) \int_{\mathcal{X}} \frac{\alpha(x, y)}{\bar{\alpha}} Y_s^n(dy) \right)^2 ds \\
&\quad + 2\mathbb{E} \left\{ \sup_{0 \leq t \leq T} [M_t^n(\phi)]^2 \right\} \\
&\stackrel{\text{def}}{=} 2\mathbb{E}\langle Y_0^n, \phi \rangle^2 + \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV} + \mathbf{V}. \tag{3.10}
\end{aligned}$$

To the end, we give estimate of every term in the equation above separately. As for term **V**, by Doob's maximal inequality and (2.9), we have that

$$\begin{aligned}
\mathbf{V} &\leq 2 \times 4\mathbb{E}[M_T^n(\phi)^2] \\
&\leq 8\mathbb{E}[\langle M^n(1) \rangle_T] \\
&\leq 8(\bar{b} + \bar{d})\mathbb{E}\left\{\int_0^T \sup_{0 \leq u \leq s} \langle X_u^n, 1 \rangle ds\right\} + 8\bar{\alpha}\mathbb{E}\int_0^T \sup_{0 \leq u \leq s} \langle X_u^n, 1 \rangle^2 ds \\
&\leq 8(\bar{b} + \bar{d} + \bar{\alpha})T \cdot (C_T^{(0)} + C_T^{(1)}),
\end{aligned} \tag{3.11}$$

where the last inequality is due to (3.6).

Since $\|\frac{b(x)}{b}\int_{\mathbb{R}^d} \phi(x+z)D(x,dz)\|_\infty \leq 1$, $\|\frac{d(x)}{d}\phi(x)\|_\infty \leq 1$, one obtains that

$$\mathbf{I} + \mathbf{II} \leq 2T^2(\bar{b}^2 + \bar{d}^2) \int_0^T \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E}[\sup_{0 \leq u \leq s} \langle Y_u^n, \phi \rangle^2] ds. \tag{3.12}$$

Similarly, **IV** can be bounded by $4T^2\bar{\alpha}^2C \int_0^T \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E}[\sup_{0 \leq u \leq s} \langle Y_u^n, \phi \rangle^2] ds$ with some constant C since $\tilde{\phi}(y) = \int_{\mathcal{X}} X_u(dx) \frac{\alpha(x,y)}{\bar{\alpha}} \phi(x)$ is supremum norm bounded by a constant due to the boundedness condition (2.6).

Term **III** is the source of all troubles. We estimate it via Lemma 3.8 to get

$$\mathbf{III} \leq 4T\bar{\alpha}^2m^d \int_0^T \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E}[\langle \phi, Y_s^n \rangle^2 \langle 1, X_s^n \rangle^2] ds.$$

For any $A > 0$ we have that

$$\begin{aligned}
\mathbf{III} &\leq 4TA^2\bar{\alpha}^2m^d \int_0^T \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E}\left[\sup_{0 \leq u \leq s} \langle \phi, Y_u^n \rangle^2\right] ds \\
&\quad + 4T\bar{\alpha}^2m^d \int_0^T \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E}[\langle \phi, Y_s^n \rangle^2 \langle 1, X_s^n \rangle^2; \langle 1, X_s^n \rangle > A] ds
\end{aligned}$$

From Lemma 3.7 we get that with some $D > 0$

$$\sup_{n \geq 1} 4TA^2\bar{\alpha}^2m^d \int_0^T \mathbb{E}[\langle \phi, Y_s^n \rangle^2 \langle 1, X_s^n \rangle^2; \langle 1, X_s^n \rangle > A] ds < D < \infty.$$

Therefore

$$\mathbf{III} \leq 4TA^2\bar{\alpha}^2m^d \int_0^T \sup_{\|\phi\|_\infty \leq 1} \mathbb{E}\left[\sup_{0 \leq u \leq s} \langle \phi, Y_u^n \rangle^2\right] ds + D$$

Let $G^n(T) := \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E}[\sup_{0 \leq t \leq T} \langle Y_t^n, \phi \rangle^2]$, by combining the estimates above and (3.10), one obtains that

$$\begin{aligned}
G^n(T) &\leq 2 \sup_{n \geq 1} \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E}\langle Y_0^n, \phi \rangle^2 + 8(\bar{b} + \bar{d} + \bar{\alpha})T \cdot (C_T^{(0)} + C_T^{(1)}) \\
&\quad + \left(2T^2(\bar{b}^2 + \bar{d}^2) + 4T^2\bar{\alpha}^2C\right) \int_0^T G^n(s) ds
\end{aligned} \tag{3.13}$$

By Gronwall's lemma, we have that

$$\begin{aligned} G^n(T) &\leq \left(2 \sup_{n \geq 1} \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} \langle Y_0^n, \phi \rangle^2 + 8(\bar{b} + \bar{d} + \bar{\alpha})T \cdot (C_T^{(0)} + C_T^{(1)}) \right) \\ &\quad \cdot \exp \left\{ \left(2T^2(\bar{b}^2 + \bar{d}^2) + 4T^2\bar{\alpha}^2 C \right) T \right\} \\ &\stackrel{\text{def}}{=} C_T^{(2)}. \end{aligned} \quad (3.14)$$

Since $C_T^{(2)}$ is a n -independent constant, the lemma follows by taking supremum over $n \in \mathbb{N}$ on both sides of the last inequality. \square

Here we present some axillary results needed for the proofs above.

Lemma 3.6. *With X_s^n defined as above we have that for any $A > 0$, $n \in \mathbb{N}_+$, $s > 0$ and $j < [A]$*

$$\mathbb{P}(\langle X_s^n, 1 \rangle > A) \leq \sum_{m=0}^{nj} e^{\left(n - \frac{m}{[A]}\right) \left(s - \ln(1 + [A] \frac{1 - 1/[A]}{1 + 1/n})\right)} \mathbb{P}\left(\langle X_0^n, 1 \rangle = \frac{m}{n}\right) + \mathbb{P}(\langle X_0^n, 1 \rangle > j) \quad (3.15)$$

Next if $A = A(s)$ and n are so big that $[A] \frac{1 - 1/[A]}{1 + 1/n} > \frac{A}{2}$, $s - \ln(1 + [A]/2) < 0$ and take $j = [\beta A]$, for any $\beta \in (0, 1)$, then

$$\mathbb{P}(\langle X_s^n, 1 \rangle > A) \leq e^{n(1-\beta)(s - \ln(1 + [A]/2))} + \mathbb{P}(\langle X_0^n, 1 \rangle > [\beta A]) \quad (3.16)$$

Finally, put $A = 2e^{s+t} - 2$, $t > 0$, to get

$$\mathbb{P}(\langle X_s^n, 1 \rangle > A) \leq e^{-n(1-\beta)t} + \mathbb{P}(\langle X_0^n, 1 \rangle > [\beta A]), \quad (3.17)$$

where clearly $t \sim \ln A$, as $A \rightarrow \infty$.

Proof. For each $n \in \mathbb{N}_+$ we use the obvious pathwise upper bound for the mass of the measure X_s^n by the total mass of the rescaled pure birth process of birth rate \bar{b} and initial random measure $\tilde{\nu}_0^n = \nu_0^n$, say $\tilde{X}_s^n = \tilde{\nu}_s^n/n$. Then clearly, for any $m > 0$

$$\begin{aligned} \mathbb{P}\left(\langle X_s^n, 1 \rangle > A \mid \langle X_0^n, 1 \rangle = \frac{m}{n}\right) &\leq \mathbb{P}\left(\langle \tilde{X}_s^n, 1 \rangle > A \mid \langle X_0^n, 1 \rangle = \frac{m}{n}\right) \\ &= \mathbb{P}\left(\langle \tilde{\nu}_s^n, 1 \rangle > [A]n \mid \langle \tilde{\nu}_0^n, 1 \rangle = m\right), \end{aligned}$$

where $[A]$ stands for the smallest integer less than A . Note that, for any $m \in \mathbb{N}_+$

$$O_{m,n}(s) = \left\{ \langle \tilde{\nu}_s^n, 1 \rangle > [A]n \mid \langle \tilde{\nu}_0^n, 1 \rangle = m \right\} = \left\{ \sum_{i=0}^{\min(0, [A]n-m)} T_i(m) \leq s \right\},$$

where $T_i(m) \sim \text{Exp}(m + i)$, $i \geq 1$, are independent random variables. Assume that $k = \min(0, [A]n - m) > 0$. Then for any $\lambda > 0$ using Markov's inequality we get

$$\mathbb{P}(O_{m,n}(s)) \leq e^{\lambda s} e^{-\lambda \sum_{i=0}^k T_i(m)} = e^{\lambda s} \prod_{i=0}^k \frac{m+i}{m+i+\lambda} = e^{\lambda s} e^{\sum_{i=0}^k \ln(1 - \frac{\lambda}{m+i+\lambda})}.$$

Using $\ln(1 - x) \leq -x$ for $0 \leq x \leq 1$ we get

$$\mathbb{P}(O_{m,n}(s)) \leq e^{\lambda s} e^{-\lambda \sum_{i=0}^k \frac{1}{m+i+\lambda}} \leq e^{\lambda s} e^{-\lambda \int_{m+1+\lambda}^{m+1+k+\lambda} \frac{1}{x} dx} = e^{\lambda(s - \ln(\frac{m+k+1+\lambda}{m+1+\lambda}))}.$$

Upon choosing $\lambda = Ck = C([A]n - m)$ for some $C > 0$ we get

$$\begin{aligned}\mathbb{P}(O_{m,n}(s)) &\leq e^{C([A]n-m)\left(s-\ln\left(\frac{(1+C)[A]n+1-m}{C[A]n+1}\right)\right)} \\ &= e^{C([A]n-m)\left(s-\ln\left(1+\frac{1-Cm/[A]n}{C+1/[A]n}\right)\right)}.\end{aligned}$$

Finally choose $C = 1/[A]$ to get

$$\mathbb{P}(O_{m,n}(s)) \leq e^{\left(n-\frac{m}{[A]}\right)\left(s-\ln\left(1+[A]\frac{1-m/[A]n}{1+1/n}\right)\right)}.$$

Note that since by definition $[A]n - m \geq 0$ we see that $1 - m/[A]n \geq 1 - 1/[A]$ and hence

$$\mathbb{P}(O_{m,n}(s)) \leq e^{\left(n-\frac{m}{[A]}\right)\left(s-\ln\left(1+[A]\frac{1-1/[A]}{1+1/n}\right)\right)}$$

Since upon conditioning on the total mass of the initial condition $\langle X_0^n, 1 \rangle$ the probability of the set $\{\langle X_s^n, 1 \rangle > A\}$ can be computed via the total probability formula. Therefore (3.15) follows and next (3.16) and (3.17) are deduced by mere substitution of the choice of A, j in (3.15). \square

The next lemma allows us to handle suitable quantities.

Lemma 3.7. *We have that for any fixed $T > 0$ there is $A > 0$ such that*

$$\sup_{n \geq 1} \sup_{0 \leq s \leq T} \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} [\langle Y_s^n, \phi \rangle^2 \langle X_s^n, 1 \rangle^2; \langle X_s^n, 1 \rangle > A] < \infty$$

Proof. Clearly by the definition of $Y_s^n = \sqrt{n}(X_s^n - X_s)$ and the fact that $\langle X_s, 1 \rangle$ is a bounded deterministic process on any finite interval we get using Hölder's inequality and $(a + b)^2 \leq 2a^2 + 2b^2$ that

$$\begin{aligned}\mathbb{E} [\langle Y_s^n, \phi \rangle^2 \langle X_s^n, 1 \rangle^2; |\langle X_s^n, 1 \rangle| > A] \\ \leq 2n (1 + \langle X_s, 1 \rangle^2) \mathbb{E} [\langle X_s^n, 1 \rangle^4; |\langle X_s^n, 1 \rangle| > A]\end{aligned}$$

By a standard formula for positive random variables we have that

$$\mathbb{E} [X^4; X > A] \leq 4 \int_A^\infty \mathbb{P}(X > y) y^3 dy + 4\mathbb{P}(X > A) A^3$$

and thus

$$\begin{aligned}\mathbb{E} [\langle Y_s^n, \phi \rangle^2 \langle X_s^n, 1 \rangle^2; |\langle X_s^n, 1 \rangle| > A] \\ \leq 8n (1 + \langle X_s, 1 \rangle^2) \left(\int_A^\infty \mathbb{P}(\langle X_s^n, 1 \rangle > y) y^3 dy + A^3 \mathbb{P}(\langle X_s^n, 1 \rangle > A) \right)\end{aligned}$$

Now for $s \geq 0$ fixed and n we use Lemma 3.6 with $y = 2(e^{s+t(y)} - 1)$, for any $y \geq A$ and $\beta \in (0, 1)$ to get

$$\begin{aligned}\mathbb{E} [\langle Y_s^n, \phi \rangle^2 \langle X_s^n, 1 \rangle^2; |\langle X_s^n, 1 \rangle| > A] \\ \leq 8n (1 + \langle X_s, 1 \rangle^2) \left(\int_A^\infty e^{-n(1-\beta)t(y)} y^3 dy + \left(2(e^{s+t(A)} - 1)\right)^3 e^{-n(1-\beta)t(A)} \right) \\ + 8n (1 + \langle X_s, 1 \rangle^2) \left(\int_A^\infty \mathbb{P}(\langle X_n^0, 1 \rangle > [\beta y]) y^3 dy \right) \\ + 8n (1 + \langle X_s, 1 \rangle^2) A^3 \mathbb{P}(\langle X_n^0, 1 \rangle > [\beta A]),\end{aligned}$$

where for convenience we either keep A or substitute it with $2(e^{s+t(A)} - 1)$.

First note that according to Lemma 3.6 we have that for A big enough and $y > A$ then $t(y) \sim \ln(y)$ uniformly for $s \in [0, T]$ and $t(y) \geq t(A) \geq \ln(1 + A/2) - T$ which shows that

$$\sup_{n \geq 1} \sup_{s \leq T} \left(8n (1 + \langle X_s, 1 \rangle^2) \left(\int_A^\infty e^{-n(1-\beta)t(y)} y^3 dy + \left(2(e^{s+t(A)} - 1) \right)^3 e^{-n(1-\beta)t(A)} \right) \right) < \infty$$

Clearly, since X_s is deterministic we have that for $A > 2\langle X_0, 1 \rangle$ and $y \geq A$

$$\mathbb{P}(\langle X_0^n, 1 \rangle > y) \leq \mathbb{P}(\langle X_0^n - X_0, 1 \rangle > y/2) \leq \frac{2^{4+\delta} \mathbb{E} \left[|\langle X_0^n - X_0, 1 \rangle|^{4+\delta} \right]}{y^{4+\delta}} \wedge 1.$$

Thus using this last inequality, for A big enough, we have that

$$\begin{aligned} & \sup_{n \geq 1} \sup_{s \leq T} \left\{ 8n (1 + \langle X_s, 1 \rangle^2) \left(\int_A^\infty \mathbb{P}(\langle X_0^n, 1 \rangle > [\beta y]) y^3 dy \right) \right\} \\ & + \sup_{n \geq 1} \sup_{s \leq T} \left\{ 8n (1 + \langle X_s, 1 \rangle^2) A^3 \mathbb{P}(\langle X_0^n, 1 \rangle > [\beta A]) \right\} < \infty \end{aligned}$$

since by (3.1) and the definition of Y_0^n ,

$$\sup_{n \geq 1} \left(n \mathbb{E} \left[|\langle X_0^n - X_0, 1 \rangle|^{4+\delta} \right] \right) \leq \sup_{n \geq 1} \left(\sup_{\|\phi\|_\infty \leq 1} \mathbb{E} |\langle Y_0^n, \phi \rangle|^{4+\delta} \right) < \infty.$$

□

The next result is the key to the estimates.

Lemma 3.8. *If for some $m \in \mathbb{N}^+$, $\alpha(x, y) = \sum_{i=1}^{m^d} f_i(x) g_i(y)$, where $f_i, g_i \in C^\infty(\mathcal{X})$ and $f_i \geq 0$, $g_i \geq 0$ for all $1 \leq i \leq m^d$ then*

$$\begin{aligned} & \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} \left[\left(\int_{\mathcal{X}} \phi(x) Y_s^n(dx) \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right)^2 \right] \\ & \leq C m^d \sup_{\phi \in \mathcal{B}(\mathcal{X}); \|\phi\|_\infty \leq 1} \mathbb{E} [\langle \phi, Y_s^n \rangle^2 \langle 1, X_s^n \rangle^2] \end{aligned} \quad (3.18)$$

Proof. The proof uses the inequality $(a_1 + \dots + a_{m^d})^2 \leq m^d (a_1^2 + \dots + a_{m^d}^2)$ which readily yields

$$\begin{aligned} & \sup_{\|\phi\|_\infty \leq 1} \mathbb{E} \left[\left(\int_{\mathcal{X}} \phi(x) Y_s^n(dx) \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right)^2 \right] \\ & \leq \sup_{\|\phi\|_\infty \leq 1} \mathbb{E} \left[\left(\sum_{i=1}^{m^d} \langle \phi f_i, Y_s^n \rangle \langle g_i, X_s^n \rangle \right)^2 \right] \\ & = FG \sup_{\|\phi\|_\infty \leq 1} \mathbb{E} \left[\left(\sum_{i=1}^m \langle \phi f_i / F, Y_s^n \rangle \langle g_i / G, X_s^n \rangle \right)^2 \right] \\ & \leq m^d FG \sup_{\|\phi\|_\infty \leq 1} \mathbb{E} [\langle \phi, Y_s^n \rangle^2 \langle 1, X_s^n \rangle^2], \end{aligned}$$

where $F := \max_{i \leq m^d} \|f_i\|_\infty$ and $G := \max_{j \leq m^d} \|g_j\|_\infty$ and we have made use of the fact that $0 \leq g_j/G \leq 1$, X_s^n is a positive random measure and $\phi f_i/F \in \mathcal{B}(\mathcal{X})$. This proves (3.18). □

Proposition 3.9. *Consider a sequence of processes $(Y_t^n)_{t \geq 0}$ in $\mathbb{D}([0, \infty), \mathcal{M}_F(\mathcal{X}))$ and Y_0^n satisfying (3.1). Then, for any $\phi \in \mathcal{B}(\mathcal{X})$, the sequence of laws of the processes $\{\langle Y_t^n, \phi \rangle; n \geq 1\}$ is tight in $\mathbb{D}([0, \infty), \mathbb{R})$.*

Proof. Since $\{\langle Y_t^n, \phi \rangle; n \geq 1\}$ is a sequence of semimartingales, we verify the tightness criteria given by Aldous [1] and Rebolledo (see, e.g., Etheridge[8, Theorem 1.17]).

For any fixed $t > 0$, $\{\langle Y_t^n, \phi \rangle; n \geq 1\}$ is tight due to Lemma 3.5.

To the end, we will prove the tightness criterion of the finite variation part (say A_t^n) and the quadratic variation of the martingale part $M_t^n(\phi)$ of $\{\langle Y_t^n, \phi \rangle; n \geq 1\}$, respectively.

For any $\varepsilon > 0$ and $T > 0$, given a sequence of stopping time τ_n bounded by T . W.O.L.G., assume $\|\phi\|_\infty \leq 1$. As for the finite variation part A_t^n of (2.8), we have

$$\begin{aligned}
& \sup_{n \geq 1} \sup_{\theta \in [0, \delta]} \mathbb{P} \left[\left| A_{\tau_n + \theta}^{(n)} - A_{\tau_n}^{(n)} \right| > \varepsilon \right] \\
& \leq \frac{1}{\varepsilon^2} \sup_{n \geq 1} \sup_{\theta \in [0, \delta]} \mathbb{E} \left(A_{\tau_n + \theta}^{(n)} - A_{\tau_n}^{(n)} \right)^2 \\
& \stackrel{\text{H\"older}}{\leq} \frac{\delta}{\varepsilon^2} \sup_{n \geq 1} \sup_{\theta \in [0, \delta]} \int_{\tau_n}^{\tau_n + \theta} \mathbb{E} \left\{ \int_{\mathcal{X}} Y_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \right. \\
& \quad - \int_{\mathcal{X}} Y_s^n(dx) d(x) \phi(x) \\
& \quad - \int_{\mathcal{X}} Y_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \\
& \quad \left. - \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s^n(dy) \right\}^2 ds.
\end{aligned} \tag{3.19}$$

We use the same estimates as before and apply Lemma 3.8 to the term before the last to easily get that

$$\begin{aligned}
& \leq \frac{2\delta\bar{b}^2}{\varepsilon^2} \cdot \sup_{n \geq 1} \int_0^T \mathbb{E} \sup_{0 \leq u \leq T} \langle Y_u^n, \frac{b(\cdot)}{\bar{b}} \int_{\mathcal{X}} \phi(\cdot+z) D(\cdot, dz) \rangle^2 ds \\
& \quad + \frac{2\delta\bar{d}^2}{\varepsilon^2} \cdot \sup_{n \geq 1} \int_0^T \mathbb{E} \sup_{0 \leq u \leq T} \langle Y_u^n, \frac{d(\cdot)}{\bar{d}} \phi(\cdot) \rangle^2 ds \\
& \quad + \frac{2\delta\bar{\alpha}^2 C}{\varepsilon^2} \cdot \left(\sup_{n \geq 1} \sup_{\|\phi\|_\infty \leq 1} \int_0^T \mathbb{E} \sup_{0 \leq u \leq T} \langle Y_u^n, \phi \rangle^2 ds + D \right) \\
& \leq \delta T C_T^{(2)} C,
\end{aligned} \tag{3.20}$$

where C changes from line to line. On the other hand, from (2.9), we have that

$$\begin{aligned}
& \sup_{n \geq 1} \sup_{\theta \in [0, \delta]} \mathbb{P} \left[\left| \langle M^n(\phi) \rangle_{\tau_n + \theta} - \langle M^n(\phi) \rangle_{\tau_n} \right| > \varepsilon \right] \\
& \leq \frac{\delta(\bar{b} + \bar{d})}{\varepsilon} \cdot \sup_{n \geq 1} \mathbb{E} \sup_{0 \leq u \leq T} \langle X_u^n, 1 \rangle ds \\
& \quad + \frac{\delta\bar{\alpha}}{\varepsilon} \cdot \sup_{n \geq 1} \mathbb{E} \sup_{0 \leq u \leq T} \langle X_s^n, 1 \rangle^2 ds \\
& \leq \delta(C_T^{(0)} + C_T^{(1)})C.
\end{aligned} \tag{3.21}$$

According to the moment estimates results in Lemma 3.4 and Lemma 3.5, both inequalities (3.19) and (3.21) can be less than ε if we take δ (which only depends on $T, \varepsilon, \|\phi\|_\infty$) small

enough, i.e.

$$\sup_{n \geq 1} \sup_{\theta \in [0, \delta]} \mathbb{P} \left[\left| A_{\tau_n + \theta}^{(n)} - A_{\tau_n}^{(n)} \right| > \varepsilon \right] < \varepsilon,$$

$$\sup_{n \geq 1} \sup_{\theta \in [0, \delta]} \mathbb{P} \left[\left| \langle M^n(\phi) \rangle_{\tau_n + \theta} - \langle M^n(\phi) \rangle_{\tau_n} \right| > \varepsilon \right] < \varepsilon,$$

which fulfil the Aldous-Rebolledo tightness condition. \square

3.2. Convergence of finite dimensional distributions. In this section, we prove a weak limit of $\{(Y_t^n)_{t \geq 0}; n \geq 1\}$ in the sense of f.d.d. convergence is a solution of some martingale problem.

Proposition 3.10. *Under the conditions given in Theorem 3.1, the finite dimensional distributions of $(Y_t^n)_{t \geq 0}$ converge as $n \rightarrow \infty$ to those of a $\mathcal{S}'(\mathcal{X})$ -valued Markov process $(Y_t)_{t \geq 0}$ satisfying that for $\phi \in \mathcal{S}(\mathbb{R}^d)$, the process*

$$\begin{aligned} M_t(\phi) := & \langle Y_t, \phi \rangle - \langle \gamma, \phi \rangle - \int_0^t \left\langle Y_s, b(\cdot) \int_{\mathbb{R}^d} \phi(\cdot + z) D(\cdot, dz) \right\rangle ds \\ & + \int_0^t \left\langle Y_s, d(\cdot) \phi(\cdot) \right\rangle ds \\ & + \int_0^t \left\langle Y_s, \int_{\mathcal{X}} \alpha(x, \cdot) \phi(x) X_s(dx) \right\rangle ds \\ & + \int_0^t \left\langle Y_s, \phi(\cdot) \int_{\mathcal{X}} \alpha(\cdot, y) X_s(dy) \right\rangle ds \end{aligned} \quad (3.22)$$

is a continuous martingale with quadratic variation

$$\begin{aligned} \langle M(\phi) \rangle_t = & \int_0^t ds \int_{\mathcal{X}} X_s(dx) b(x) \int_{\mathbb{R}^d} \phi^2(x + z) D(x, dz) \\ & + \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi^2(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \right). \end{aligned} \quad (3.23)$$

Proof. By Proposition 3.9, we already proved $\{(Y_t^n, \phi); n \geq 1\}$ is tight in $\mathbb{D}([0, \infty), \mathbb{R})$ for any $\phi \in \mathcal{S}(\mathbb{R}^d)$. Following Mitoma [19] (see e.g., Ethier and Kurtz [10, Theorem 3.9.1]), we conclude that the sequence $\{(Y_t^n)_{t \geq 0}; n \geq 1\}$ is tight in $\mathbb{D}([0, \infty), \mathcal{S}'(\mathbb{R}^d))$. Hence, we can assume there exists a weak limit $(Y_t)_{t \geq 0}$ of a subsequence of $\{(Y_t^n)_{t \geq 0}; n \geq 1\}$. Since $Y_t^n \in \mathcal{M}_F(\mathcal{X})$, then $\langle Y_t^n, \phi \rangle = 0$ for any $\phi \in \mathcal{S}(\mathcal{X}^c)$. Therefore, we have $Y_t \in \mathcal{S}'(\mathcal{X})$.

Firstly, we check that $(Y_t)_{t \geq 0}$ is a.s. continuous. By the construction of (Y_t^n) , we have

$$\begin{aligned} \sup_{t \in [0, T]} \sup_{\|\phi\| \leq 1} |\langle Y_t^n, \phi \rangle - \langle Y_{t-}^n, \phi \rangle| & \leq \sup_{t \in [0, T]} \sup_{\|\phi\| \leq 1} \sqrt{n} \{ |\langle X_t^n, \phi \rangle - \langle X_{t-}^n, \phi \rangle| + |\langle X_t - X_{t-}, \phi \rangle| \} \\ & \leq \sqrt{n} \frac{1}{n} + 0 \\ & = \frac{1}{\sqrt{n}}. \end{aligned} \quad (3.24)$$

By letting $n \rightarrow \infty$, it implies the continuity of $(Y_t)_{t \geq 0}$, i.e. $(Y_t)_{t \geq 0} \in \mathbb{C}([0, +\infty), \mathcal{S}'(\mathcal{X}))$.

To prove $(M_t(\phi))_{t \geq 0}$ is a martingale, it suffices to prove that

$$\mathbb{E}[M_t(\phi)] = 0. \quad (3.25)$$

Let

$$\begin{aligned}
\widetilde{M}_t^n(\phi) &:= \langle Y_t^n, \phi \rangle - \langle Y_0^n, \phi \rangle - \int_0^t ds \int_{\mathcal{X}} Y_s^n(dx) b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \\
&\quad + \int_0^t ds \int_{\mathcal{X}} d(x) \phi(x) Y_s^n(dx) \\
&\quad + \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s^n(dy) \\
&\quad + \int_0^t ds \int_{\mathcal{X}} Y_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s(dy).
\end{aligned} \tag{3.26}$$

Then, for fixed $t > 0$ and any $n \in \mathbb{N}$, we have

$$|\mathbb{E}[M_t(\phi)]| \leq |\mathbb{E}[M_t^n(\phi) - \widetilde{M}_t^n(\phi)]| + |\mathbb{E}[\widetilde{M}_t^n(\phi) - M_t(\phi)]| + |\mathbb{E}[M_t^n(\phi)]|. \tag{3.27}$$

According to Proposition 2.2, we have $\mathbb{E}[M_t^n(\phi)] = 0$.

Since $\{(Y_t^n)_{t \geq 0}; n \geq 1\}$ converges in law to $(Y_t)_{t \geq 0}$ as $n \rightarrow \infty$ and $(\widetilde{M}_t^n(\phi) - M_t(\phi))$ is homogeneous w.r.t. $(Y_t^n - Y_t)$, we get

$$\lim_{n \rightarrow \infty} |\mathbb{E}[\widetilde{M}_t^n(\phi) - M_t(\phi)]| = 0. \tag{3.28}$$

As for the first term on RHS of (3.27),

$$\begin{aligned}
&|\mathbb{E}[M_t^n(\phi) - \widetilde{M}_t^n(\phi)]| \\
&= \left| \mathbb{E} \left\{ \sqrt{n} \int_0^t ds \int_{\mathcal{X}} \left(\frac{Y_s^n(dx)}{\sqrt{n}} + X_s(dx) \right) \phi(x) \int_{\mathcal{X}} \alpha(x, y) \left(\frac{Y_s^n(dy)}{\sqrt{n}} + X_s(dy) \right) \right. \right. \\
&\quad - \sqrt{n} \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \\
&\quad - \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s^n(dy) \\
&\quad \left. \left. - \int_0^t ds \int_{\mathcal{X}} Y_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \right\} \right| \\
&\leq \frac{1}{\sqrt{n}} \left| \mathbb{E} \int_0^t ds \int_{\mathcal{X}} Y_s^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s^n(dy) \right| \\
&\leq \frac{1}{\sqrt{n}} \bar{\alpha} T \|\phi\|_{\infty} C_t^{(2)} \\
&\xrightarrow{n \rightarrow \infty} 0,
\end{aligned} \tag{3.29}$$

where $C_t^{(2)}$ is determined as in Lemma 3.5.

By combining the above estimates together, we conclude $|\mathbb{E}[M_t(\phi)]| = 0$.

In the remainder, we will justify that the quadratic variation of $M_t(\phi)$ has the form (3.23). By applying Itô's formula to $\langle Y_t, \phi \rangle^2$, according to the semimartingale decomposition (3.22) of

$\langle Y_t, \phi \rangle$, we have

$$\begin{aligned}
\langle Y_t, \phi \rangle^2 &= \langle \gamma, \phi \rangle^2 + 2 \int_0^t \langle Y_s, \phi \rangle d[\langle Y_s, \phi \rangle] + \langle M.(\phi) \rangle_t \\
&= \langle \gamma, \phi \rangle^2 + \langle M.(\phi) \rangle_t \\
&\quad + 2 \int_0^t \langle Y_s, \phi \rangle ds \left\{ \int_{\mathcal{X}} Y_s(dx) b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - \int_{\mathcal{X}} Y_s(dx) d(x) \phi(x) \right. \\
&\quad \left. - \int_{\mathcal{X}} Y_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s(dy) - \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s(dy) \right\} \\
&\quad + \text{martingale}.
\end{aligned} \tag{3.30}$$

On the other hand, according to the definition of (Y_t^n) , we have

$$\begin{aligned}
\langle Y_t^n, \phi \rangle^2 &= \langle \sqrt{n}(X_t^n - X_t), \phi \rangle^2 \\
&= n \left[\langle X_t^n, \phi \rangle^2 - 2 \langle X_t^n, \phi \rangle \langle X_t, \phi \rangle + \langle X_t, \phi \rangle^2 \right].
\end{aligned} \tag{3.31}$$

To simplify the computations, let us introduce new notation:

$$\begin{aligned}
A(s) &:= \int_{\mathcal{X}} X_s(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - \phi(x) (d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s(dy)) \right], \\
B^n(s) &:= \int_{\mathcal{X}} X_s^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - \phi(x) (d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy)) \right].
\end{aligned} \tag{3.32}$$

From (2.10), (2.11) and (2.5), respectively, it follows that

$$\begin{aligned}
\langle X_t^n, \phi \rangle &= \langle X_0^n, \phi \rangle + \int_0^t B^n(s) ds + \text{martingale}, \\
\langle X_t^n, \phi \rangle^2 &= \langle X_0^n, \phi \rangle^2 + 2 \int_0^t \langle X_s^n, \phi \rangle B^n(s) ds + \frac{1}{n} \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) \\
&\quad \left[b(x) \int_{\mathbb{R}^d} \phi^2(x+z) D(x, dz) + \phi^2(x) (d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy)) \right] + \text{martingale}, \\
\langle X_t, \phi \rangle &= \langle X_0, \phi \rangle + \int_0^t A(s) ds.
\end{aligned} \tag{3.33}$$

By substituting every term above into (3.31), we have that

$$\begin{aligned}
\langle Y_t^n, \phi \rangle^2 &= n \langle X_0^n, \phi \rangle^2 + \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi^2(x+z) D(x, dz) \right. \\
&\quad \left. + \phi^2(x) (d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy)) \right] + 2n \int_0^t \langle X_s^n, \phi \rangle B^n(s) ds \\
&\quad - 2n \left[\langle X_0^n, \phi \rangle + \int_0^t B^n(s) ds \right] \left[\langle X_0, \phi \rangle + \int_0^t A(s) ds \right] \\
&\quad + n \left[\langle X_0, \phi \rangle + \int_0^t A(s) ds \right]^2 + \text{martingale}.
\end{aligned} \tag{3.34}$$

Set

$$D^{t,n,1} := \int_0^t ds \int_{\mathcal{X}} X_s^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi^2(x+z) D(x, dz) + \phi^2(x) (d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy)) \right]. \tag{3.35}$$

By combining all the quadratic terms at time 0 in (3.34) together, it follows that

$$\begin{aligned}
(3.34) = & n\langle X_0^n - X_0, \phi \rangle^2 + D^{t,n,1} \\
& + 2n \int_0^t \langle X_s^n, \phi \rangle B^n(s) ds - 2n\langle X_t, \phi \rangle \int_0^t B^n(s) ds \\
& - 2n\langle X_0^n, \phi \rangle \int_0^t A(s) ds + 2n\langle X_0, \phi \rangle \int_0^t A(s) ds \\
& + n \left[\int_0^t A(s) ds \right]^2 + \text{martingale}
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
= & \langle Y_0^n, \phi \rangle^2 + D^{t,n,1} \\
& + 2n \int_0^t \left\langle \frac{Y_s^n}{\sqrt{n}} + X_s, \phi \right\rangle B^n(s) ds - 2n\langle X_t, \phi \rangle \int_0^t B^n(s) ds \\
& - 2\sqrt{n}\langle Y_0^n, \phi \rangle \int_0^t A(s) ds + n \left[\int_0^t A(s) ds \right]^2 + \text{martingale}
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
= & \langle Y_0^n, \phi \rangle^2 + D^{t,n,1} \\
& + 2\sqrt{n} \int_0^t \langle Y_s^n, \phi \rangle B^n(s) ds \\
& + 2n \int_0^t \langle X_s, \phi \rangle B^n(s) ds - 2n\langle X_t, \phi \rangle \int_0^t B^n(s) ds \\
& - 2\sqrt{n}\langle Y_0^n, \phi \rangle \int_0^t A(s) ds + n \left[\int_0^t A(s) ds \right]^2 + \text{martingale}
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
& \stackrel{\text{Integration by parts}}{=} \langle Y_0^n, \phi \rangle^2 + D^{t,n,1} \\
& + 2\sqrt{n} \int_0^t \langle Y_s^n, \phi \rangle B^n(s) ds \\
& - 2n \int_0^t ds A(s) \int_0^s B^n(r) dr - 2\sqrt{n}\langle Y_0^n, \phi \rangle \int_0^t A(s) ds \\
& + n \left[\int_0^t A(s) ds \right]^2 + \text{martingale}
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
& \text{Replace } B^n(s) \text{ by (3.32)} \quad \langle Y_0^n, \phi \rangle^2 + D^{t,n,1} \\
& + 2 \int_0^t \langle Y_s^n, \phi \rangle ds \left\{ \int_{\mathcal{X}} Y_s^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - \phi(x) \right] \right. \\
& \quad \left. + \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right\} - \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s^n(dy) \Big\} \\
& + 2\sqrt{n} \int_0^t \langle Y_s^n, \phi \rangle A(s) ds \\
& - 2n \int_0^t ds A(s) \int_0^s B^n(r) dr - 2\sqrt{n}\langle Y_0^n, \phi \rangle \int_0^t A(s) ds \\
& + n \left[\int_0^t A(s) ds \right]^2 + \text{martingale}.
\end{aligned} \tag{3.40}$$

Set

$$D^{t,n,2} := 2 \int_0^t \langle Y_s^n, \phi \rangle ds \left\{ \int_{\mathcal{X}} Y_s^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - d(x) \phi(x) \right. \right. \\ \left. \left. - \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s^n(dy) \right] - \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s^n(dy) \right\}. \quad (3.41)$$

Replacing $\langle Y_s^n, \phi \rangle$ and $B^n(r)$ by (2.8) and (3.32) respectively, one obtains that

$$(3.40) = \langle Y_0^n, \phi \rangle^2 + D^{t,n,1} + D^{t,n,2} \\ + 2\sqrt{n} \int_0^t ds A(s) \int_0^s dr \int_{\mathcal{X}} Y_r^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - d(x) \phi(x) \right] \\ - 2n \int_0^t ds A(s) \int_0^s dr \int_{\mathcal{X}} X_r^n(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_r^n(dy) \\ + 2n \int_0^t ds A(s) \int_0^s dr \int_{\mathcal{X}} X_r(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_r(dy) \\ - 2n \int_0^t ds A(s) \int_0^s dr \int_{\mathcal{X}} X_r^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) \right. \\ \left. - \phi(x) (d(x) + \alpha(x, y) X_r^n(dy)) \right] + n \left[\int_0^t A(s) ds \right]^2 + \text{martingale} \quad (3.42)$$

$$= \langle Y_0^n, \phi \rangle^2 + D^{t,n,1} + D^{t,n,2} \\ + 2\sqrt{n} \int_0^t ds A(s) \int_0^s dr \int_{\mathcal{X}} Y_r^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - d(x) \phi(x) \right] \\ + 2n \int_0^t ds A(s) \int_0^s dr \int_{\mathcal{X}} X_r(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_r(dy) \\ - 2n \int_0^t ds A(s) \int_0^s dr \int_{\mathcal{X}} X_r^n(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - d(x) \phi(x) \right] \\ + n \left[\int_0^t A(s) ds \right]^2 + \text{martingale}. \quad (3.43)$$

Recombining X_r^n , X_r and Y_r^n , it thus follows

$$\langle Y_t^n, \phi \rangle^2 = \langle Y_0^n, \phi \rangle^2 + D^{t,n,1} + D^{t,n,2} \\ - 2n \int_0^t ds A(s) \int_0^s A(r) dr \\ + n \left[\int_0^t A(s) ds \right]^2 + \text{martingale} \quad (3.44)$$

$$\stackrel{\text{Integration by parts}}{=} \langle Y_0^n, \phi \rangle^2 + D^{t,n,1} + D^{t,n,2} + \text{martingale}.$$

Obviously, both $D^{t,n,1}$ and $D^{t,n,2}$ converge as $n \rightarrow \infty$.

Finally, we get that

$$\begin{aligned}
\langle Y_t, \phi \rangle^2 &= \langle Y_0, \phi \rangle^2 + \int_0^t ds \int_{\mathcal{X}} X_s(dx) b(x) \int_{\mathbb{R}^d} \phi^2(x+z) D(x, dz) \\
&\quad + \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi^2(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \right) \\
&\quad + 2 \int_0^t \langle Y_s, \phi \rangle ds \left\{ \int_{\mathcal{X}} Y_s(dx) \left[b(x) \int_{\mathbb{R}^d} \phi(x+z) D(x, dz) - d(x) \phi(x) \right. \right. \\
&\quad \left. \left. - \phi(x) \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \right] - \int_{\mathcal{X}} X_s(dx) \phi(x) \int_{\mathcal{X}} \alpha(x, y) Y_s(dy) \right\} \\
&\quad + \text{martingale.}
\end{aligned} \tag{3.45}$$

By comparing the representations of (3.30) and (3.45), we conclude that

$$\begin{aligned}
\langle M_*(\phi) \rangle_t &= \int_0^t ds \int_{\mathcal{X}} X_s(dx) b(x) \int_{\mathbb{R}^d} \phi^2(x+z) D(x, dz) \\
&\quad + \int_0^t ds \int_{\mathcal{X}} X_s(dx) \phi^2(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y) X_s(dy) \right).
\end{aligned} \tag{3.46}$$

□

3.3. Uniqueness of the martingale problem. Instead of proving the uniqueness of the solution of the limiting martingale problem directly, we associate it with the solution of a corresponding generalized Langevin equation which will be shown in the next section. We will show that the Langevin equation has a unique solution (see Theorem 4.3).

4. LINKS WITH GENERALIZED LANGEVIN EQUATIONS

A criterion for an infinite-dimensional Gaussian process (distribution-valued process) to satisfy a generalized Langevin equation is given in [3], where both of the evolution term and the white noise term are time inhomogeneous. In this section, we apply the criterion to our fluctuation limit obtained in the previous section.

Definition 4.1. An $\mathcal{S}'(\mathbb{R}^d)$ -valued process $\{W_t; t \in \mathbb{R}^+\}$ is called (centered) Gaussian if $\{\langle W_t, \phi \rangle; t \in \mathbb{R}^+, \phi \in \mathcal{S}(\mathbb{R}^d)\}$ is a (centered) Gaussian system. We say that $\{W_t; t \in \mathbb{R}^+\}$ is a $\mathcal{S}'(\mathcal{X})$ -valued process if $\langle W_t, \phi \rangle \equiv 0$, for any $\phi \in \mathcal{S}(\mathcal{X}^c)$.

Definition 4.2. A centered Gaussian $\mathcal{S}'(\mathbb{R}^d)$ -valued process $W = \{W_t; t \in \mathbb{R}^+\}$ is called a generalized Wiener process if it has continuous paths and its covariance functional $C(s, \phi; t, \psi) := \mathbb{E}[\langle W_s, \phi \rangle \langle W_t, \psi \rangle]$ has the form

$$C(s, \phi; t, \psi) = \int_0^{s \wedge t} \langle Q_u \phi, \psi \rangle du, \quad s, t \in \mathbb{R}^+, \phi, \psi \in \mathcal{S}(\mathbb{R}^d), \tag{4.1}$$

where the operators $Q_u : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ have the following properties:

- (1) Q_u is linear, continuous, symmetric and positive for each $u \in \mathbb{R}^+$,
- (2) the function $u \rightarrow \langle Q_u \phi, \psi \rangle$ is right continuous with left limit for each $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$.

We then say that W is associated to Q .

Let's remind that we inherit the same notation as in Section 2 and Section 3. Define $Q_t\phi \in \mathcal{S}'(\mathbb{R}^d)$ for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $t \in \mathbb{R}^+$ by

$$\begin{aligned} \langle Q_t\phi, \psi \rangle := \int_{\mathcal{X}} X_t(dx) & \left[b(x) \int_{\mathcal{X}-x} \phi(x+z)\psi(x+z)D(x, dz) \right. \\ & \left. + \phi(x)\psi(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y)X_t(dy) \right) \right], \quad \text{for } \psi \in \mathcal{S}(\mathbb{R}^d). \end{aligned} \quad (4.2)$$

Recall the quadratic variation form of $M_t(\phi)$ in (3.3). It follows from a direct fact that $\langle M_t(\phi) \rangle_t = \int_0^t \langle Q_u\phi, \phi \rangle du$. Then, we have

Theorem 4.3. *The fluctuation limit process $(Y_t)_{t \geq 0}$ obtained in Theorem 3.1 is the unique solution of a time inhomogeneous Langevin equation*

$$\begin{cases} dY_t = A_t^* Y_t dt + dW_t, & t > 0 \\ Y_0 = \gamma \end{cases}, \quad (4.3)$$

where A_t^* denotes the adjoint operator of A_t defined by

$$\begin{aligned} A_t\phi(x) = b(x) \int_{\mathcal{X}} \phi(x+z)D(x, dz) - \phi(x) \left(d(x) + \int_{\mathcal{X}} \alpha(x, y)X_t(dy) \right) \\ - \int_{\mathcal{X}} \alpha(y, x)\phi(y)X_t(dy), \end{aligned} \quad (4.4)$$

and $(W_t)_{t \geq 0}$ is an $\mathcal{S}'(\mathcal{X})$ -valued Wiener process with covariance

$$\mathbb{E}[\langle W_s, \phi \rangle \langle W_t, \psi \rangle] = \int_0^{s \wedge t} \langle Q_u\phi, \psi \rangle du, \quad s, t \geq 0, \quad \phi, \psi \in \mathcal{S}(\mathbb{R}^d). \quad (4.5)$$

Remark 4.4. (1) An $\mathcal{S}'(\mathbb{R}^d)$ -valued process $(Y_t)_{t \geq 0}$ is said to be a solution of (4.3) if for each $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle Y_t, \phi \rangle = \langle \gamma, \phi \rangle + \int_0^t \langle Y_u, A_u\phi \rangle du + \langle W_t, \phi \rangle, \quad \text{for } t \in \mathbb{R}^+. \quad (4.6)$$

(2) $(W_t)_{t \geq 0}$ has independent increments but not the stationary property since the covariance functional Q depends on the time.

Proof. Existence. According to Theorem 3.1, the covariance functional of the continuous martingale M_t on testing functions is deterministic, which implies that $(M_t)_{t \geq 0}$ is a $\mathcal{S}'(\mathbb{R}^d)$ -valued mean zero Gaussian process (see Walsh[20, Proposition 2.10]). Hence, $(Y_t)_{t \geq 0}$ is also an $\mathcal{S}'(\mathbb{R}^d)$ -valued Gaussian process.

Set $K(s, \phi; t, \psi) := \mathbb{E}[\langle Y_s, \phi \rangle \langle Y_t, \psi \rangle]$. To the end, one needs eventually to show that

$$\frac{\partial}{\partial t} K(t, \phi; t, \psi) - K(t, A_t\phi; t, \psi) - K(t, \phi; t, A_t\psi) = \langle Q_t\phi, \psi \rangle. \quad (4.7)$$

By applying Itô formula to $\langle Y_s, \phi \rangle \langle Y_t, \psi \rangle$ and (3.2), we have that

$$\langle Y_t, \phi \rangle \langle Y_t, \psi \rangle = \int_0^t \langle Y_u, \phi \rangle d\langle Y_u, \psi \rangle + \int_0^t \langle Y_u, \psi \rangle d\langle Y_u, \phi \rangle + \int_0^t d\langle Y_u, \psi \rangle d\langle Y_u, \phi \rangle. \quad (4.8)$$

Then taking expectation on both sides, we obtain that

$$K(t, \phi; t, \psi) = \int_0^t \langle Y_u, \phi \rangle \langle Y_u, A_u\psi \rangle du + \int_0^t \langle Y_u, \psi \rangle \langle Y_u, A_u\phi \rangle du + \int_0^t d[M_t(\psi), M_t(\phi)]_u. \quad (4.9)$$

Differentiate the last equation with respect to t , we conclude

$$\begin{aligned} \frac{\partial}{\partial t} K(t, \phi; t, \psi) &= K(t, A_t \phi; t, \psi) + K(t, \phi; t, A_t \psi) + \frac{\partial}{\partial t} \mathbb{E}[M_t(\psi) M_t(\phi)] \\ &= K(t, A_t \phi; t, \psi) + K(t, \phi; t, A_t \psi) + \langle Q_t \phi, \psi \rangle, \end{aligned} \quad (4.10)$$

where the last equality is due to (3.3) and (4.2).

On the other hand, it is not hard to check that $(Q_t)_{t \geq 0}$ satisfies the conditions required in Definition 4.2. Finally, by the results of [3, Theorem 2], there exists an $\mathcal{S}'(\mathbb{R}^d)$ -valued Wiener process $(W_t)_{t \geq 0}$ associated to the covariance functional $(Q_t)_{t \geq 0}$ such that $(Y_t)_{t \geq 0}$ satisfies the generalized Langevin equation (4.3) driven by a generalized Wiener process $(W_t)_{t \geq 0}$. It remains to show that moreover $(W_t)_{t \geq 0}$ is in fact $\mathcal{S}'(\mathcal{X})$ -valued problem. Note that from (4.2) we have that if $\phi \in \mathcal{S}(\mathcal{X}^c)$ or $\psi \in \mathcal{S}(\mathcal{X}^c)$ then $\langle Q_t \phi, \psi \rangle = 0$, for any $t \geq 0$. This is due to the definition of $b(x)$, $d(x)$, $\alpha(x, y)$ and $D(x, dz)$. Thus in (4.1) $C(s, \phi; s, \phi) = \mathbb{E}[\langle W_s, \phi \rangle^2] = 0$ and therefore since $(W_t)_{t \geq 0}$ is centered we conclude that $\langle W_s, \phi \rangle \equiv 0$.

Uniqueness. First note that Assumption (A1) implies easily that $A_t \phi \in \mathcal{S}(\mathbb{R}^d)$ for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ since \mathcal{X} and henceforth $\mathcal{X} - \mathcal{X}$ are compact and any differentiation of (4.4) can be taken under the integrals of the right-hand side of (4.4). Indeed all terms but $\int_{\mathcal{X}} \phi(x+z) m(x, z) dz$ are obvious. However since $m(x, z) \in C^\infty(\mathcal{X} \times (\mathcal{X} - \mathcal{X}))$ then $\sup_{x \in \mathcal{X}, z \in \mathcal{X} - \mathcal{X}} \left| \frac{d^n}{dx^n} m(x, z) \right| < \infty$ for any $n \geq 0$ we conclude that $\int_{\mathcal{X}} \phi(x+z) m(x, z) dz \in \mathcal{S}(\mathbb{R}^d)$.

Since all coefficients are bounded, the linear operator $A_t : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R}^d)$ is a supremum norm uniformly bounded for $t \in [0, T]$ for any $T > 0$. Therefore, the equation (4.3) has a unique $\mathcal{S}'(\mathbb{R}^d)$ -valued solution given by the mild form:

$$Y_t = T_{0,t}^* \gamma + \int_0^t T_{r,t}^* dW_r, \quad (4.11)$$

where $\{T_{r,t} : 0 \leq r \leq t < +\infty\}$ is the unique reversed evolution system generated by $(A_t)_{t \geq 0}$ and $T_{r,t}^*$ is its adjoint operator of $T_{r,t}$. We refer the reader to [13, Theorem 2.1] for details. \square

5. ONE DIMENSIONAL TIME-INHOMOGENEOUS ORNSTEIN-UHLENBECK PROCESS

In this section, we will study a degenerate case as an example of Theorem 4.3. Consider the case when there is no spatial dispersal and all the individuals stay at the same position, i.e. $D(x, dz) = 1_{\{z=0\}}$ in dispersal kernel (2.1).

Proposition 5.1. *Admit the same conditions as in Theorem 3.1. In particular, assume $X_t^n = \xi_t^n \delta_x$ and $D(x, dz) = 1_{\{z=0\}}$ in (2.1). Then, $(\xi_t^n, \eta_t^n)_{t \geq 0}$ converge in law to $(\xi_t, \eta_t)_{t \geq 0}$ as $n \rightarrow \infty$ which satisfies the following equations:*

$$\begin{cases} d\xi_t = (b(x) - d(x) - \alpha(x, x)\xi_t)\xi_t dt \\ d\eta_t = (b(x) - d(x) - 2\alpha(x, x)\xi_t)\eta_t dt + \sqrt{(b(x) + d(x) + \alpha(x, x)\xi_t)}\xi_t dB_t, \end{cases} \quad (5.1)$$

where $\eta_t^n := \sqrt{n}(\xi_t^n - \xi_t)$.

Remark 5.2. *We can regard the system above as an inhomogeneous Ornstein-Uhlenbeck (OU) process living in a deterministic environment. We refer the reader to [10, Theorem 11.2.3] for a general limiting process defined by a one-dim inhomogeneous Langevin equation.*

Proof. Since $D(x, dz) = 1_{\{z=0\}}$, by taking $\phi = 1$ in (2.5), we can easily show that there exists a process $(\xi_t)_{t \geq 0}$ defined by $\xi_t := \langle X_t, 1 \rangle$ solving the first equation in (5.1). Taking $\phi = 1$, from (4.4), we have that

$$\langle Y_t, A_t 1 \rangle = (b(x) - d(x) - 2\alpha(x, x)\xi_t)\langle Y_t, 1 \rangle. \quad (5.2)$$

From (4.2) and (4.5), we have that

$$\mathbb{E}\langle W_t, 1 \rangle^2 = \int_0^t (b(x) + d(x) + \alpha(x, x)\xi_s)\xi_s ds. \quad (5.3)$$

Define

$$B_t = \int_0^t \left[(b(x) + d(x) + \alpha(x, x)\xi_s)\xi_s \right]^{-\frac{1}{2}} d\langle W_s, 1 \rangle. \quad (5.4)$$

Then, we get its quadratic variation $\langle B \rangle_t = t$. Thus, $(B_t)_{t \geq 0}$ is a standard Brownian motion. Furthermore, we have

$$d\langle W_t, 1 \rangle = \sqrt{(b(x) + d(x) + \alpha(x, x)\xi_t)\xi_t} \cdot dB_t. \quad (5.5)$$

Let $\eta_t := \langle Y_t, 1 \rangle$, by taking (5.2) and (5.5) back to (4.6) when $\phi = 1$, the second equation in (5.1) follows. \square

In the next result, we study the stationary distribution of the system (5.1).

Proposition 5.3. *Suppose the process $(\eta_t)_{t \geq 0}$ is defined as in (5.1). Then, it has a stationary distribution which is Gaussian $\mathcal{N}(0, \frac{b(x)}{\alpha(x, x)})$.*

Remark 5.4. *When we consider the long term behavior, as long as $b(x) > d(x)$, it always has the same fluctuation no matter which value the death rate $d(x)$ takes.*

Proof. Let

$$\begin{aligned} \theta_t &:= -(b(x) - d(x) - 2\alpha(x, x)\xi_t), \\ \sigma_t &:= \sqrt{(b(x) + d(x) + \alpha(x, x)\xi_t)\xi_t}. \end{aligned}$$

From (5.1), it follows that

$$d\eta_t = -\theta_t \eta_t dt + \sigma_t dB_t. \quad (5.6)$$

The characteristic function of $(\eta_t)_{t \geq 0}$ has the form

$$\mathbb{E}_{\eta_0} [e^{iz\eta_t}] = \exp \left\{ iz e^{-\int_0^t \theta_u du} \eta_0 - \frac{1}{2} z^2 \int_0^t \sigma_u^2 e^{-2\int_0^u \theta_v dv} du \right\}. \quad (5.7)$$

Since ξ_t in (5.1) has a unique stable equilibrium $\frac{b(x)-d(x)}{\alpha(x, x)}$, it follows that $\lim_{t \rightarrow \infty} \theta_t = b(x) - d(x) > 0$ and $\lim_{t \rightarrow \infty} \sigma_t^2 = 2b(x)(b(x) - d(x))/\alpha(x, x)$.

Then,

$$\begin{aligned} \lim_{t \rightarrow \infty} \log \mathbb{E}_{\eta_0} [e^{iz\eta_t}] &= - \lim_{t \rightarrow \infty} \frac{1}{2} z^2 \cdot \frac{\int_0^t \sigma_u^2 e^{2\int_0^u \theta_v dv} du}{e^{2\int_0^t \theta_u du}} \\ &= - \frac{1}{2} z^2 \cdot \lim_{t \rightarrow \infty} \frac{\sigma_t^2}{2\theta_t} \\ &= - \frac{1}{2} z^2 \frac{b(x)}{\alpha(x, x)}. \end{aligned} \quad (5.8)$$

Finally, we conclude that $(\eta_t)_{t \geq 0}$ has stationary distribution $\mathcal{N}(0, \frac{b(x)}{\alpha(x, x)})$. \square

REFERENCES

- [1] ALDOUS, D. (1978). Stopping times and tightness. *Ann. Probab.* **6**, 335-340.
- [2] BOLKER, B. AND PACALA, S. (1997). Using moment equations to understand stochastically driven spatial pattern formation in ecological systems. *Theor. Popul. Biol.* **52**, 179-197.
- [3] BOJDECKI, T. AND GOROSTIZA, L.G. (1988). Inhomogeneous infinite dimensional langevin equations, *Stoch. Anal. Appl.* **6(1)**, 1-9.
- [4] CHAPAGNAT, N. (2006). A microscopic interpretation for adaptive dynamics trait substitution sequence models. *Stoch. Proc. Appl.* **116**, 1127-1160.
- [5] CHAMPAGNAT, N. AND MÉLÉARD, S. (2011). Polymorphic evolution sequence and evolutionary branching. *Probab. Theory Relat. Fields.* **151**, 45-94
- [6] DAWSON, D.A. (1993). *Measure-valued Markov processes*. *École d'Été de Probabilités de Saint Flour XXI*. Lecture Notes in Mathematics **1541**, 1-261. Springer-Verlag.
- [7] DAWSON, D.A. AND GREVEN, A. (2010). Multiscale analysis: Fisher-Wright diffusions with rare mutations and selection, logistic branching system. arXiv:1007.5462v1
- [8] ETHERIDGE, A.M. (2000). *An introduction to superprocesses*. Univ. Lecture Series **20**. AMS, Rhode Island.
- [9] ETHERIDGE, A.M. (2004). Survival and extinction in a locally regulated population. *Ann. Appl. Probab.* **14**, 188-214.
- [10] Ethier, S.N. and Kurtz, T.G. (1986). *Markov processes: characterization and convergence*. John Wiley and Sons, New York.
- [11] FOURNIER, N. AND MÉLÉARD, S. (2004) A microscopic probabilistic description of a locally regulated population and macroscopic approximation. *Ann. Appl. Probab.* **14**, 1880-1919.””
- [12] GOROSTIZA, L.G. LI, Z.H. (1999). High density fluctuations of immigration branching particle systems. *Stochastic Models, CMS conference proceedings*. **26**, 159-171.
- [13] KALLIANPUR, G., PEREZ-ABREU, V. (1988). Stochastic evolution equations driven by nuclear space valued martingales. *Appl. Math. Optim.* **17**, 237-272.
- [14] Kurtz, T. G. (1971). Limit theorems for sequences of jump markov processes approximating ordinary differential processes. *J. Appl. Probab.* **8**, 344-356.
- [15] LAMBERT A. (2005). The branching process with logistic growth. *Ann. Appl. Prob.* **15**, 1506-1535.
- [16] LAW, R. AND DIECKMANN, U. (2002). Moment approximations of individual-based models. *The Geometry of Ecological Interactions: Simplifying Spatial Complexity*. (eds. Dieckmann, U., Law, R., Metz, J.A.J.) 252-270. Cambridge Univ. Press.
- [17] LI, Z.H. (2011). *Measure-valued branching Markov processes*. Springer.
- [18] MÉLÉARD, S. (1997). Convergence of the fluctuations for the interacting diffusions with jumps associated with Boltzmann equations *Stoch. and Stoch. Reports* **63**, 195-225.
- [19] MITOMA, I. (1983). Tightness of probabilities on $\mathbb{C}([0, 1], S')$ and $\mathbb{D}([0, 1], S')$. *Ann. Probab.* **11**, 989-999.
- [20] WALSH, J.B. (1986). An introduction to stochastic partial differential equations. *Lect. Notes Math.* **1180**. Springer-Verlag.

M. SAVOV, INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES, 15 NOEMVRI STR., 1040 - SOFIA, BULGARIA

E-mail address: m.savov@reading.ac.uk

S.-D. WANG, CORRESPONDING AUTHOR: DEPARTMENT OF STATISTICS, UNIVERSITY OF OXFORD, 1 SOUTH PARKS ROAD, OXFORD, OX1 3TG, UK

E-mail address: shidong.wang@stats.ox.ac.uk